On some quantitative aspects of categorical Lagrangian Topology

Giovanni Ambrosioni

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From physics to geometry

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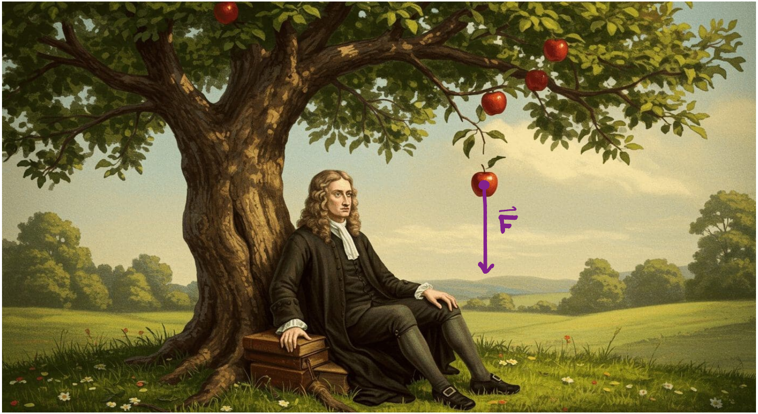

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- Often the force acting on the system arises from a *potential energy function* $U \colon N \to \mathbb{R}$.
- We write the above as

$$\begin{cases} m\ddot{q}(t) = -\nabla_q U(q(t)) \\ (q(0), p(0)) = (q_0, p_0) \end{cases}$$

for initial conditions (q_0, p_0).

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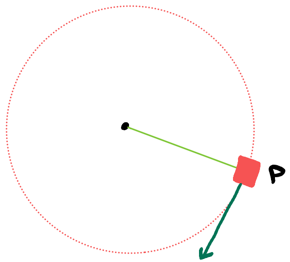
  "All possible positions $\times$ all possible momenta."
- For instance, the pendulum:
  It can move on a circle S^1, and the configuration space is

$$T^*S^1 \cong S^1 \times \mathbb{R}$$

  (velocity can be arbitrarily large, and in two directions).

From physics to geometry



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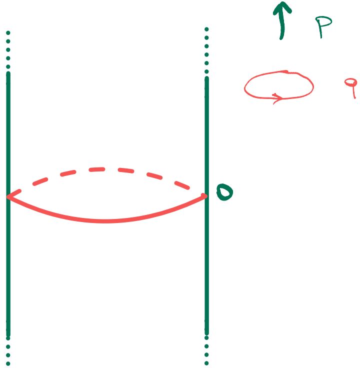
$(q(t), p(t)) \in T^*S^1 \cong S^1 \times \mathbb{R}$



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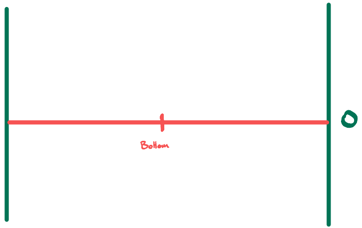

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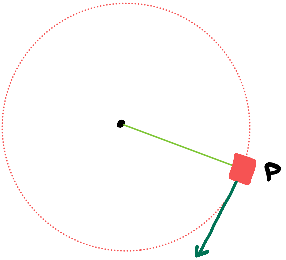

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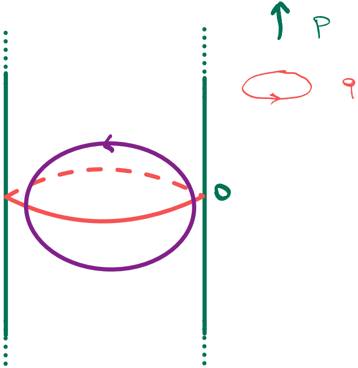

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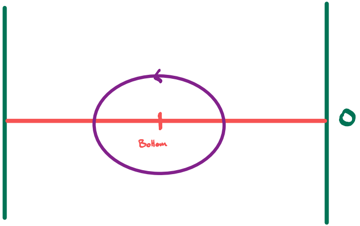

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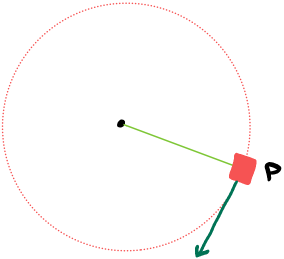

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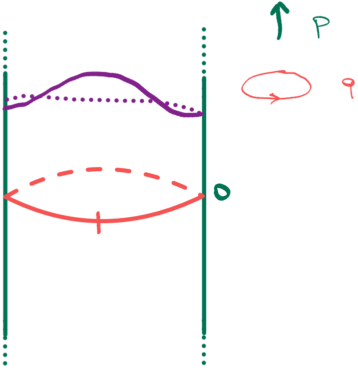

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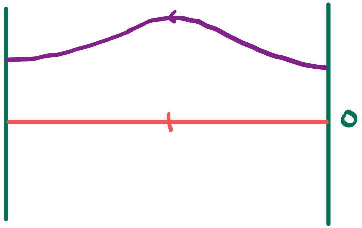

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- Mathematically:

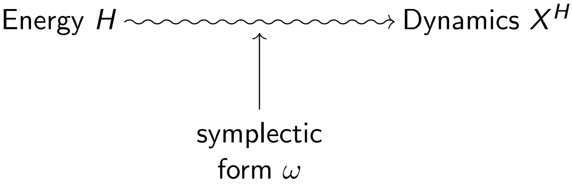
$$\begin{gathered}\text{Configuration space } q(t) \in N \\ m\ddot{q}(t) = -\nabla_q U(q(t)) \\ \rightsquigarrow \\ \text{Phase space } (q(t), p(t)) \in T^*N \\ \dot{q}(t) = \partial_p H(q,p) \\ \dot{p}(t) = -\partial_q H(q,p)\end{gathered}$$

From physics to geometry

$$\text{Energy } H \rightsquigarrow \text{Dynamics } X^H$$

From physics to geometry

Abstracting:



From physics to geometry

Abstracting:

$$\text{Energy } H \rightsquigarrow \text{Dynamics } X^H$$

$\uparrow$

symplectic
form ω

Sym-plectic $\cong$ com-plex $\cong$ intertwined (position and momentum).

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  2. Can we start here and end up there?

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Fix $q_0, q_1 \in \mathbb{R}^3$.

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$$F_{q_0} = \{q_0\} \times \mathbb{R}^3 \subset T^*\mathbb{R}^3 \text{ and } F_{q_1} = \{q_1\} \times \mathbb{R}^3 \subset T^*\mathbb{R}^3$$

and the Hamiltonian flow φ^H induced by the energy H. Does there exist a trajectory (q, p) connecting F_{q_0} to F_{q_1}?

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$\Rightarrow$ Interesection of Lagrangians.

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- There is a natural notion of isotopy: $H \colon M \times [0,1] \to \mathbb{R}$ induces a vector field X_H via

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- Let $L, N \subset (M, \omega)$ be Lagrangians. We say that L is *Hamiltonian isotopic* to N, $L \sim N$, if

$$\exists \varphi \in \mathrm{Ham}(M) : \quad L = \varphi(N).$$

Preliminaries

We are interested in studying

$$\mathcal{L}\text{ag} := \mathcal{L}\text{ag}^*(M, \omega)$$

a class of Lagrangians (with decorations) satisfying some additional assumptions $*$.

Why?

1. Hamiltonian mechanics (see above),
2. Mirror symmetry,
3. "Everything is a Lagrangian"

...

The classification of elements of $\mathcal{L}\text{ag}$ is a very difficult task.

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- $\mathcal{L}\text{ag} := \{\text{closed, exact Lagrangians } (L, h_L) \text{ contained in } D_1^*N\}$ where

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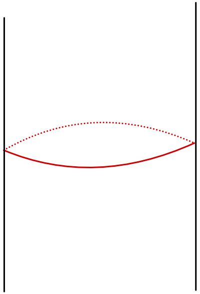
Examples of exact Lagrangians:

- zero-section $N \subset T^*N$,
- graphs $(\mathrm{graph}(df), f)$,
- non-closed example: cotangent fibers $(F_x = T_x^*N, 0)$

$$T^*S^1 \simeq S^1 \times \mathbb{R}$$



zero-section $S^1 \times \{0\}$

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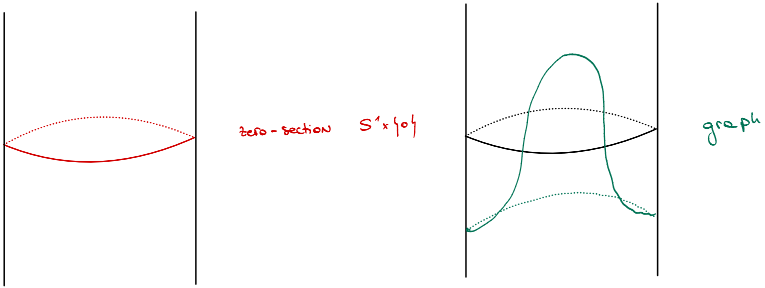

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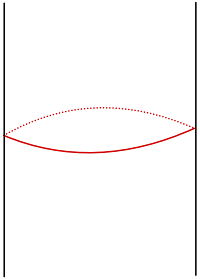

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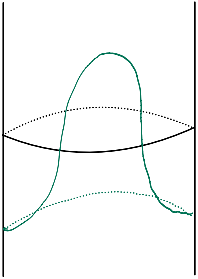
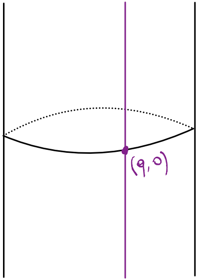


graph



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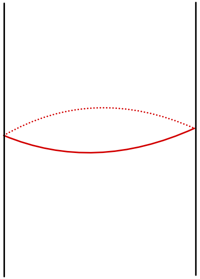

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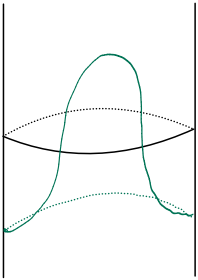

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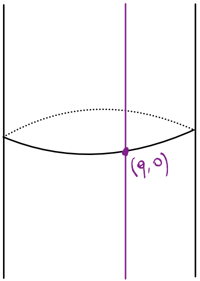

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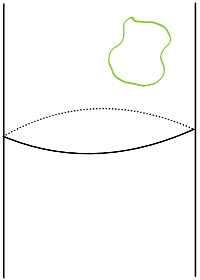

non-exact.

Conjecture (Arnold) Let $L \in \mathcal{L}\mathrm{ag}$ be Hamiltonian isotopic to N, assume $L \pitchfork N$. Then

$$|L \cap N| \geq \sum_i \dim H_i(N, \mathbb{Z}_2).$$

"Lags cannot get much wilder than graphs from intersection POV"

Conjecture (ANLC). Any Lagrangian in $\mathcal{L}\mathrm{ag}$ is Hamiltonian isotopic to the zero-section N.

"There are no exotic Lagrangians"

Conjecture (Viterbo) When endowed with the Lagrangian spectral metric γ, $(\mathcal{L}\mathrm{ag}, \gamma)$ is bounded.

"$\mathcal{L}\mathrm{ag}$ is not very large."

Global qualitative
Lagrangian Floer homology
potentials, GW-inv.

Global quantitative
Floer persistence module
spectral inv., displ. energy
Gromov width

Universal qualitative
Donaldson category

Universal quantitative
Metrics on $\mathcal{L}$ag

Compositional qualitative
Triangulated categories
(Fukaya category, cobordism,
symplectic A_∞-bicategory)

Compositional quantitative
Quantitative categorical
Lagrangian topology

The idea

Metrics on $\mathcal{L}ag$

study/
get new

Triangulated categories
(Fukaya category, cobordism,
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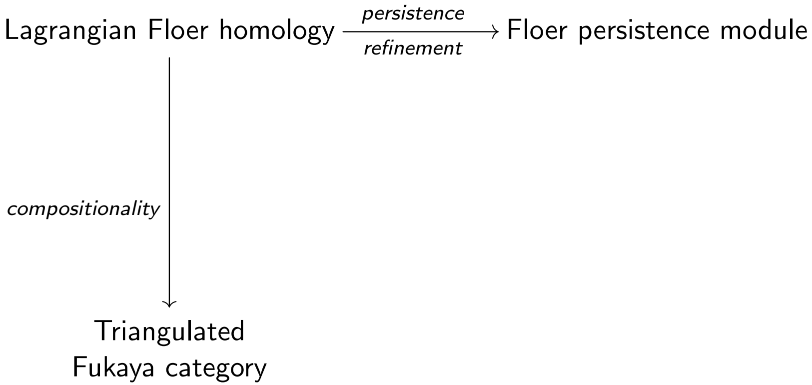
quantitative
enrichment

Quantitative categorical
Lagrangian topology

What we are able to do today:



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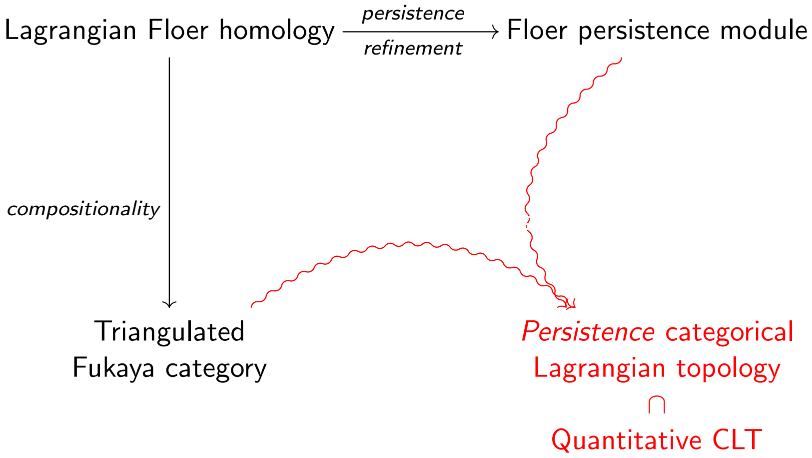

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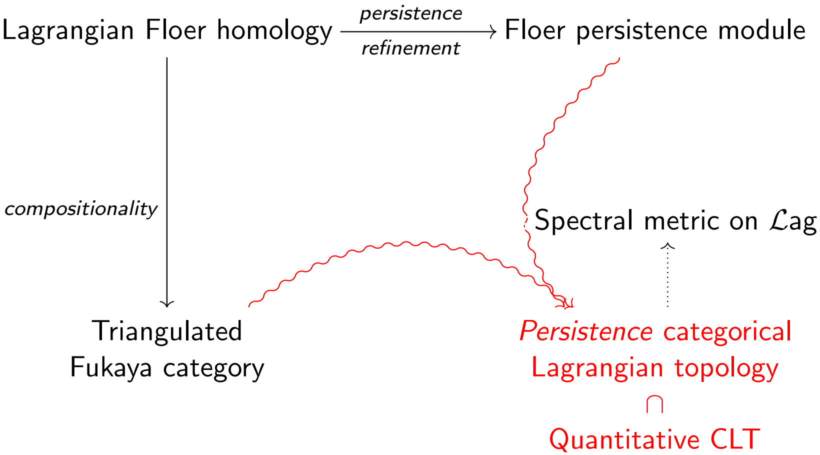


The basic tool: Lagrangian Floer homology

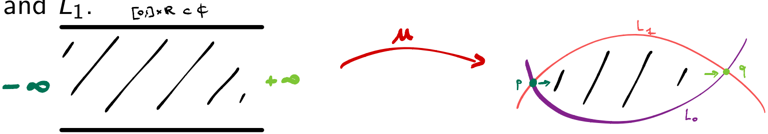
Let $L_0, L_1 \in \mathcal{L}ag$ such that $L_0 \pitchfork L_1$.

$$CF(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 \cdot p$$

Let J be an ω-compatible (Gromov '85) almost complex structure:

$$dp := \sum_{y \in L_0 \cap L_1} |\mathcal{M}(p, q, J)|_2 \cdot q$$

i.e. we "count (rigid) J-holomorphic strips" with boundary on L_0 and L_1.



Theorem (Floer '88). $(CF(L_0, L_1), d)$ is a chain complex. Moreover, if L_0 is Hamiltonian isotopic to L_1, then

$$HF(L_0, L_1) := H(CF(L_0, L_1)) \cong H(L_0).$$

Corollary. The Arnold conjecture.

Lagrangian Floer homology $\xrightarrow[\textit{refinement}]{\textit{persistence}}$ Floer persistence modules

compositionality $\downarrow$

(triangulated)
categorical structures
(Fukaya categories, cobordism,
symplectic A_∞-bicategory)

The Fukaya category (Fukaya '93, FOOO, Seidel '06)

We organise $\mathcal{L}$ag into the class of objects of an A_∞-*category* Fuk($\mathcal{L}$ag), by working **at the chain level**.

- Objects: $\mathrm{Obj}(\mathrm{Fuk}(\mathcal{L}\mathrm{ag})) := \mathcal{L}\mathrm{ag}$

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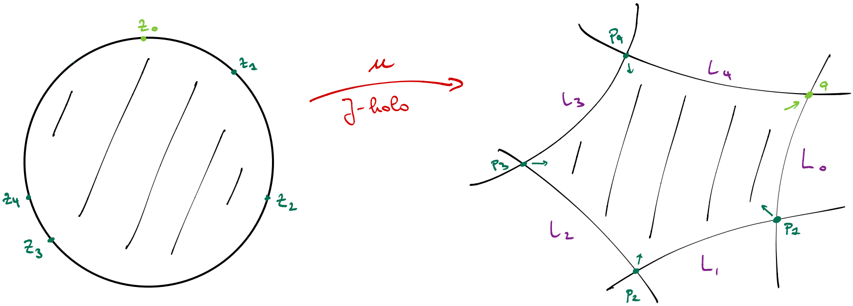
  $\hom_{\mathrm{Fuk}(\mathcal{L}\mathrm{ag})}(L_0, L_1) := CF(L_0, L_1)$,
- For any $d \geq 1$

$$\mu_d \colon CF(L_0, L_1) \otimes \cdots \otimes CF(L_{d-1}, L_d) \to CF(L_0, L_d)$$

  count J-holomorphic **polygons** with boundary on the L_i's.

The Fukaya category



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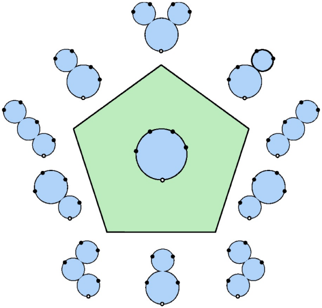

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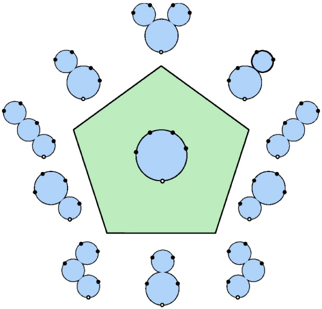

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- Read: an higher algebraic structure controlling associativity reflecting the recursive structure of the moduli space of domains.

The Fukaya category

In a dry language:

$$\mu_1 \circ \mu_d + \sum_{\substack{i+j=d-1 \\ i,j \geq 0}} \mu_d(id^i \otimes \mu_1 \otimes \mathrm{id}^j) = \sum_{\substack{i+j+k=d+1 \\ i,j\geq 0,\ k \geq 1}} \mu_{i+j+1}(id^i \otimes \mu_k \otimes id^j)$$

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The Fukaya category

In a dry language:

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- $\mu_2(a, \mu_2(b, c)) + \mu_2(\mu_2(a, b), c) =$
  $\mu_1\mu_3(a, b, c) + \mu_3(\mu_1 a, b, c) + \mu_3(a, \mu_1 b, c) + \mu_3(a, b, \mu_1 c)$

The Fukaya category

Computing Fuk($\mathcal{L}$ag) is an extremely difficult task, but:

The rich A_∞-structure often allows us to understand Fuk($\mathcal{L}$ag) in terms of a small enough set of objects.

We have to enlarge Fuk($\mathcal{L}$ag) *algebraically*. "Paradoxically", this enlargment is easier to understand.

The derived Fukaya category

- An A_∞-module is a couple $\Big((\mathcal{M}(L))_{L\in\mathcal{L}\mathrm{ag}}, (\mu^{\mathcal{M}}_{d|1})_{d\geq 0}\Big)$.

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- We can define cones of morphisms of A_∞-modules:

$$f\colon \mathcal{M} \to \mathcal{N} \Rightarrow \mathrm{Cone}(f) := \left(\mathcal{N} \oplus \mathcal{M}[1], \begin{pmatrix} \mu^{\mathcal{M}} & f \\ 0 & \mu^{\mathcal{N}} \end{pmatrix}\right)$$

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  Cones are the homological tool to build/decompose objects.
- Yoneda A_∞-functor embedding

$$\mathcal{Y}\colon \mathrm{Fuk}(\mathcal{L}\mathsf{ag}) \to A_\infty\mathrm{mod}_{\mathrm{Fuk}(\mathcal{L}\mathsf{ag})}$$

The derived Fukaya category

$\mathrm{Fuk}(\mathcal{L}\mathrm{ag})$

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Theorem
$D\mathrm{Fuk}(\mathcal{L}ag)$ is a triangulated category

The derived Fukaya category

- $\mathrm{Obj}(\mathrm{DFuk}(\mathcal{L}\mathrm{ag})) = \mathcal{L}\mathrm{ag} \cup \{\mathrm{fibers}\} \cup \{\text{iterated cones}\}$.

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- We understand some exact triangles geometrically: Dehn twists (Seidel), Lagrangian surgery (FOOO), Lagrangian cobordism (Biran-Cornea).
- In some sense, $\mathrm{Obj}(\mathrm{DFuk}(\mathcal{L}\mathrm{ag}))$ = immersed Lagrangians (Kontsevich, Biran-Cornea).

Generation

In our case, we get that $\mathcal{L}ag \subset \mathrm{Obj}(\mathrm{DFuk}(\mathcal{L}ag))$ is "very simple":

Theorem (Abouzaid '12)

For any $L \in \mathcal{L}ag$ there is a finite iterated cone in $DFuk(\mathcal{L}ag)$

$$C_L := icone(F \to \cdots \to F)$$

where F is a fiber, such that $L \cong C_L$.

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Theorem (Abouzaid '12)

In $DFuk(\mathcal{L}ag)$, all objects of $\mathcal{L}ag$ are isomorphic. In particular, for any $L_0, L_1 \in \mathcal{L}ag$ we have $HF(L_0, L_1) \cong H(N)$.

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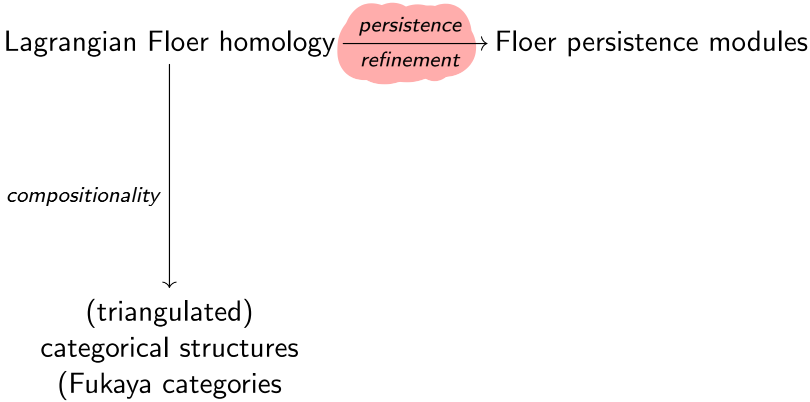
- How would a quantitative analogue of the previous generation theorem look like?
- Is any Lagrangian $L \in \mathcal{L}\mathrm{ag}$ "close" to such a cone?

Until now, we did not care much about primitives. More structure is available.



Persistence structures

Given exact Lagrangians (L_0, h_0) and (L_1, h_1), define

$$\mathbb{A}\colon L_0 \cap L_1 \to \mathbb{R},\ x \mapsto h_{L_1}(x) - h_{L_1}(x)$$

the *action* of x.
It is easy to see that $\mathbb{A}$ defines an increasing filtration of $CF(L_0, L_1)$, i.e. for any $\alpha \in \mathbb{R}$,

$$CF^{\leq\alpha}(L_0, L_1) := \bigoplus_{x\colon\ \mathbb{A}(x)\leq\alpha} \mathbb{Z}_2 \cdot x$$

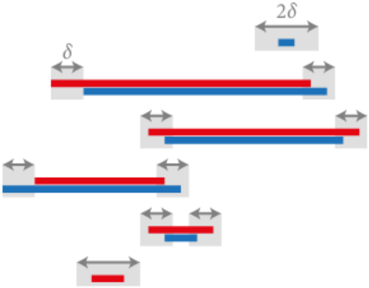
is a subcomplex of $CF(L_0, L_1)$. In modern terminology:

$HF(L_0, L_1)$ inherits the structure of a persistence module.

The interleaving distance

There is a notion of distance between two persistence modules V and W, called *interleaving distance* $d_{\text{int}}(V, W)$.

There is a beautiful interpretation of it in terms of matching of barcodes (Chasal et al.): let $\delta > 0$, a δ-matching between two barcodes (red and blue) is



$d_{\text{int}}(V, W) = \inf \{\delta > 0 : \ \exists \mu : \mathcal{B}(V) \to \mathcal{B}(W) \ \delta - \text{matching}\}$.

Note $d_{\text{int}}(V, W) < \infty \Leftrightarrow V_\infty \cong W_\infty$.

Persistence structures

We define a (pseudo-)metric γ on $\mathcal{L}$ag by looking at the difference between:

- The startpoint of the "last" infinite bar in $HF(L_0, L_1)$
- The startpoint of the "first" infinite bar in $HF(L_0, L_1)$

Conjecture (Viterbo). The diameter of $(\mathcal{L}\text{ag}, \gamma)$ is finite

Proved by Shelukin in 2020 in a few cases with ad-hoc methods.

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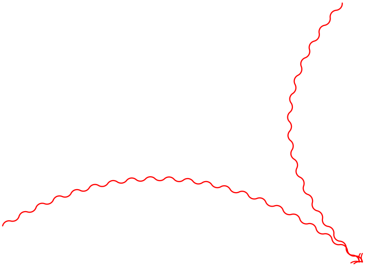
Floer persistence module



Triangulated
Fukaya category

Persistence categorical
Lagrangian topology

Issues:

1. What is the (an?) appropriate "quantitative categorical" setting algebraically?

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2. How to develop a Fukaya-like language in this setting?

3. What does it give us? What notion of "complicated"?

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A persistence category is a category $\mathcal{C}$ enriched over the (symmetric monoidal) category of persistence modules.

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$\mathcal{C}$ induces two (additive) categories with the same objects:

- $\mathcal{C}_0$, with morphisms $\hom_{\mathcal{C}_0}(X, Y) := \hom_{\mathcal{C}}(X, Y)_0$

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The enrichment over **Pmod** induces an *interleaving distance* $d_{\text{int}}^{\mathcal{C}}$ on $\text{Ob}(\mathcal{C})$.

A triangulated persistence category **TPC** is a persistence category $\mathcal{C}$ such that $\mathcal{C}_0$ is a triangulated category (+...).

- Fact: C_∞ is naturally triangulated.
- Fundamental example: homotopy category of filtered chain complexes (we gain: "weighted" acyclicity).

In this world, the first step towards

Triangulated categories
(Fukaya categories, cobordism, symplectic A_∞-bicategory) $\underset{\textit{enrichment}}{\overset{\textit{quantitative}}{\rightsquigarrow}}$ Quantitative categorical Lagrangian topology

amounts to:

Find a TPC $\mathcal{C}$ such that $\mathcal{C}_\infty \cong \mathrm{DFuk}(\mathcal{L}ag)$.

2. Filtered Fukaya categories (A. '23)

This can be achieved:

Theorem (A. '23)
There is a model for a Fukaya TPC.

1. Construct "a" filtered Fukaya A_∞-category (that is μ_d-maps preserve the filtration & units at zero level):
   - Use a "cluster" model for Floer "polygons" (Cornea-Lalonde, Charest),
   - Construct a particular class of monotone homotopies for perturbations.
2. Via persistence-abstract-nonsense (BCZ) get a TPC.

- We call this the *persistence derived Fukaya category* $PD\mathrm{Fuk}(\mathcal{L}\mathrm{ag})$.

More precisely:

Theorem (A. '23)

There is a homotopy system of filtered and strictly unital A_∞-categories with increasing accuracy

$$\left(Fuk(\mathcal{L}ag; p)_{p\in\mathcal{P}}, \left(\mathcal{H}^h_{p,q} \colon Fuk(\mathcal{L}ag; p) \to Fuk(\mathcal{L}ag; q)\right)_{\substack{p,q\in\mathcal{P}, \\ h\in\mathcal{J}_{p,q}}} \right)$$

where each A_∞-category $Fuk(\mathcal{L}ag; p)$, $p \in \mathcal{P}$, is quasi-equivalent to the standard model from the Fukaya category of $\mathcal{L}ag$.

Moreover:

Lemma (BCZ, Guillermou-Viterbo)

The metric d_{int} on the Fukaya TPC extends γ, i.e.

$$d_{int}|_{\mathcal{L}ag} \approx \gamma.$$

- That is, we extended the definition of a symplectically meaningful metric on $\mathcal{L}$ag defined via persistence to the objects of a triangulated category, which we know being "well"-generated.

3. Approximability (A.-Biran-Cornea '26)

Theorem
The space $(\mathcal{L}ag, \gamma)$ *is* **approximable** *in* $DFuk(\mathcal{L}ag)$ *via its TPC-refinement* $PDFuk(\mathcal{L}ag)$*, that is*

$$\forall \varepsilon > 0, \exists \mathcal{F}_\varepsilon \subset \{fibers\} \text{ finite}$$

such that

$$\forall L \in \mathcal{L}ag,\ \exists C_L = icone(F_1 \to \cdots \to F_{k(L,\varepsilon)}) \text{ in } PDFuk(\mathcal{L}ag)_0$$

where $F_i \in \mathcal{F}_\varepsilon$ *such that*

$$d_{int}(L, C_L) < \varepsilon.$$

Approximability

Abstractly speaking, approximability can be seen both as a generalization of

- generation in triangulated categories (take $\varepsilon \to \infty$)
- total-boundedness ($k(L, \varepsilon)$) independent of L).
- a concept of approximability for metric groups introduced by Turing (indeed $\mathcal{L}\text{ag} \to K_0(\text{PDFuk}(\mathcal{L}\text{ag}))$ is approximable in this sense).

Approximability might be the right notion on which to build to tackle questions about the metric structure of $(\mathcal{L}\text{ag}, \gamma)$. Moreover, it allows to consider dynamical problems in symplectic topology from a categorical viewpoint.

Approximability and categorical entropy

Let $\varepsilon > 0$, φ be a self map of $(\mathcal{L}\mathrm{ag}, \gamma)$, and $L \in \mathcal{L}\mathrm{ag}$, define

$$h(\varphi, L, \varepsilon) := \lim_{n\to\infty} \frac{\log(k(\varphi^n L, \varepsilon))}{n}$$

(Dimitrov-Haiden-Katzarkov-Kontsevich for $\varepsilon \to \infty$.)

Theorem (ABC '25)

In some cotangent bundles, there are examples of $L \in \mathcal{L}ag$ and φ induced by Hamiltonian diffeomorphisms such that, for small enough ε we have $h(\varphi, L, \varepsilon) > 0$.

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> That is, persistence categorical structures are sensitive to dynamics, whereas the classical triangulated is not.

Approximability and categorical entropy

Corollary (ABC '25)

There is N closed such that the associated $(\mathcal{L}ag, \gamma)$ is not totally bounded.

- Equivalently, its completion is not compact.
- Hence, it is not possible to define topological entropy directly.
- Note that this does not disprove Viterbo's conjecture.
- More on this next week in the Symplectic seminar.

Similar result hold in other cases, where approximability is replaced by *retract*-approximability, for instance:

- Equators on S^2 are *retract*-approximated by great circles passing thorugh north and south poles,
- Non-contractible circles on $\mathbb{T}^2$ are *retract*-approximated by one longitude and straight latitudes.

In these cases, we have a geometric *retract-approximability criterion.*

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  - Towards TWC?
  - "Adjusted" shadow metric from Lagrangian cobordism to approximate d_H?

Grazie!